Singular Value Decomposition (SVD) as an Approach for Digital Image Compression

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***Abstract*—The problem that we tried to solve is the processing of digital images collected by photographing agricultural land from the air. Images created in this way are most often used in the process of assessing the condition of sown crops on productive agricultural land. In order to obtain high quality images, anemone aircraft equipped with high-resolution cameras are in use. High-resolution photos usually require a large storage space, so we must work on their compression, which must not affect the quality of the photos themselves. Singular Value Decomposition has taken a significant place in the implementation of image processing applications in recent years. It's importance is especially reflected in the fact that it provides quality compression of images, and object recognition on images. In this paper, we propose a survey for the SVD as an efficient transform method, which can be used in compression of various types of images, and thus images created for the needs of smart agricultural production.**

***Index Terms*—SVD; Image compression; Object recognition.**

I. Introduction

THE domain of precision agriculture is reflected in the use of modern information and communication solutions in order to improve productivity by making decisions based on a set of data dating from the past [1]. Recording crops on agricultural land both from the air and from the ground is one of the most commonly used methods for data collection. In this way, depending on the hardware equipment and skills of the operator, photos and videos of different quality can be obtained. As information extraction depends on the quality of the image, recorded images often have to be processed. In addition, high-resolution images require a large amount of storage space, and it is often necessary to compress the image using various digital image processing methods.

Since the digital image data can be represented in the matrix form, the digital image processing methods can utilize a number of mathematical techniques. The essential subject areas are computational linear algebra, integral transforms, statistics and other techniques of numerical analysis. Various digital image processing algorithms can be written in term of matrix equation, hence, computational method in linear algebra come to be an important aspect of the subject [2]. Digital image processing encompasses a wide and varied field of application, such as area of image operation and compression, computer vision, and image analysis (called image understanding). There is the consideration of three types of computerized processing: low level processing where both its inputs and outputs are images; mid-level processing where inputs are images, but outputs are attributes extracted from those images, and higher-level processing that involves "making sense" of an ensemble of recognized objects as in image analysis, and performing the cognitive function associated with human vision [2].

One of the methods that can be used in digital image processing is Singular Value Decomposition (SVD), and this method represents a highlight of linear algebra [3]. More precisely in linear algebra, the SVD of a matrix is a factorization of that matrix into three matrices. It has some interesting algebraic properties and conveys important geometrical and theoretical insights about linear transformations. It also has some important applications in data science. The technique of SVD has a long and somewhat surprising history. It started out in the social sciences with intelligence testing. SVD is known under many different names. In the early days it was called "factor analysis." Other terms include principal component (PC) decomposition and empirical orthogonal function (EOF) analysis. All these are mathematically equivalent, although the way they are treated in the literature is often quite different [4]. Today, singular value decomposition has spread through many branches of science, in particular psychology and sociology, climate and atmospheric science, image processing and astronomy. It is also extremely useful in machine learning and in both descriptive and predictive statistics. From the aspect of digital image processing, image features can be divided in four main groups. Those four groups are visual features, statistical pixel features, transform coefficient features, and algebraic features. Based on those four groups, SVD technique can be observed as an algebraic feature.

The paper is organized as fallows. Second section represents literature review. Third section represents theory of SVD. Fourth section represents application of SVD in image processing, and the fifth section represents main conclusions.

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II. LITERATURE REVIEW

Research in the field of application of SVD method in image processing began in the 70's of the 20th century [5,6]. In one of the researches authors reviewed potential application of SVD in various aspects of digital image processing. They conclude that the easiest way to understand the use of SVD is to realize that a sampled image is nothing more than an array of scalar values and therefore an equivalent to a matrix. The theory of SVD is that of representing matrices (and therefore images) as sums of orthogonal matrices of rank one (outer products) [7]. The decomposition of SVD of an image into a space with diagonal representation then allows implementation of simple enhancement procedures using scalar linear or nonlinear filtering. Such filters suggest the existence of a family of images diagonal in the space of eigenimages of the original picture.

Authors in [8] had applied theory of linear algebra to digital image processing. The aim of the research was the application of SVD within two areas of digital image processing: image compression and face recognition. These two specific areas of digital image processing were firstly investigated and then the application of SVD in these image processing areas is tested. Various experiments with different singular value were performed, and the compression result was evaluated by compression ratio and quality measurement. To perform face recognition with SVD, authors treated the set of known faces as vectors in a subspace, called "face space", spanned by a small group of "base-faces". The projection of a new image onto the base-face was then compared with the set of known faces to identify the face. Authors used MATLAB as computing environment and programming language for the purpose of implementation and execution of all tests and experiments. Based on the theory and result of experiments, authors found that SVD is a stable and effective method to split the system into a set of linearly independent components, where each of them is carrying own data (information) to contribute to the system, Thus, both rank of the problem and subspace orientation can be determined. Overall, the SVD approach is robust, simple, easy and fast to implement. It works well in a constrained environment. It provides a practical solution to image compression and recognition problem. Image compression was the topic of interest in one more research. Beside image compression, authors in this research illustrate the use of SVD on matrix completion [9]. The former was to convert the original full-rank pixel matrix to a well-approximated low-rank matrix and thus dramatically save the space. After that, authors recover a pixel matrix with a large number of missing entries by using nuclear norm minimization, in which some singular value thresholding algorithm is used. For both applications, authors conduct numerical experiments to show the performance and point out some possible improvements in the future. Authors conclude that SVD used in image compression process effectively reduces the size of bitmap images. Reducing the image size leads to significant savings in memory space. As a result, the difference between the image restored from the approximated SVD and the original one is negligible and undistinguishable by human eyes. The second application was created in order to recover digital images in the absence of data due to various reasons. The recovery of the full data set from the observed ones largely resembles other related challenges. In order to reduce the use of computing power, authors used algorithm that first identifies the active subspaces and subsequently applies SVD to a matrix that is much smaller in size for each iteration. The authors managed to reconstruct the image with as many as 75% of missing data points.

Authors in [10] show a novel technique for wavelet-based corner detection using singular value decomposition. In presented approach, SVD facilitates the selection of global natural scale in discrete wavelet transform. Authors define natural scale as the level associated with most prominent (dominant) eigenvalue. Created eigenvector corresponding to dominant eigenvalue is considered as the natural scale. The corners are detected at the locations corresponding to modulus maxima. This technique has the advantage of analyzing the wavelet decomposition at natural scale. Algorithm is given for the selection of natural wavelet scale under discrete wavelet domain.

Authors in [11] describe one of the approaches, which integrate both singular value decomposition of each image to increase the compactness density distribution and hybrid color space suitable to this case constituted by the three relevant chromatics levels deduced by histogram analysis. More precisely, their proposition describes the efficiency of SVD and color information to subtract background pixels corresponding to shadows pixels. Singular value decomposition is used in order to increase the compactness power of each distribution. In the same time SVD approximate a new image, which is manipulated, and determine a suitable color space constituted by significant levels among set of levels commonly used in color image analysis. Furthermore, this technique proves that the gray levels and the RGB space are not efficient for all applications. However, the deduced color space (Hrb) shows convincing results and confirm the efficiency of this method.

III. THEORY OF SINGULAR VALUE DECOMPOSITION

In this section, a mathematical description of the SVD method is given which is needed in order to understand what this method represents, its implementation and its application to image compression problems.

The decomposition of the matrix into singular values is a factorization that occurs as a step in many algorithms of applied linear algebra. It is equally important in a conceptual sense because it describes the properties of a factorized matrix. Singular values of a matrix represent a generalization of the concept of eigenvalues of square matrices and exist for an arbitrary square or rectangular matrix. This type of factorization overcomes the problems encountered in the process of diagonalization of square matrices, and on that occasion retains the most important properties of orthogonal diagonalization of symmetric matrices [12]

The starting point of the SVD procedure is that an arbitrary rectangular matrix $A \in M_{mxn}$ represents a linear mapping of an n-dimensional vector space into an m-dimensional space $A: R^n \to R^m$, or $A: C^n \to C^m$ in a complex case. The diagonalization of square matrices is related to the choice of one base of vector space that facilitates calculations. Rectangular matrices act over different spaces. It is a natural question whether two orthonormal bases of these two vector spaces can be found, which will describe its properties as transformations through the simplest form of matrix *A*.

A matrix of size $m \times n$ is a grid of real numbers consisting of m rows and n columns. When we have an (m × n)-matrix A and a (n × k)-matrix B, we can compute the product AB which is an (m × k)-matrix. The mapping corresponding to AB is exactly the composition of the mappings corresponding to A and B respectively. Singular Value Decomposition (SVD) states that every (m × n)-matrix A can be written as a product. a graphical representation of the creation of the matrix a is shown in Fig. 1.

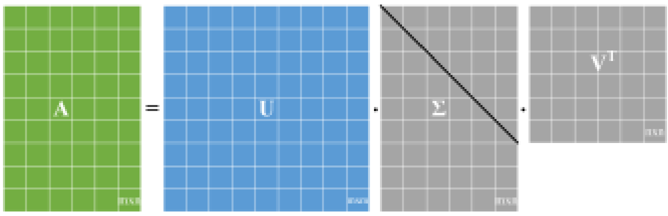

Fig. 1. Construction of matrix A

In this product $U \in M_{mxm}$ and $V \in M_{nxn}$ are orthogonal matrices and the matrix $\Sigma \in M_{mxn}$ consists of descending non-negative values on its diagonal and zeros elsewhere, so it is valid

$$A = U\Sigma V^T \tag{1}$$

The diagonal elements σ_i of the matrix Σ are called the singular values of the matrix A. The matrix Σ can be shown in following:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \sigma_{r+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{2}$$

The columns of the matrix U are called the left singular vectors, and the columns of the matrix V are the right singular vectors. It is a consequence of equality, as in

$$AV = U\Sigma \text{ and } U^T A = \Sigma V^T \tag{3}$$

$$Av_i = \sigma_i u_i \quad u_i^T A = \sigma_i v_i^T \quad i = 1,2,\ldots,\min\{m,n\} \quad (4)$$

Equation 1 can be viewed as a transformation of a unit sphere of n-dimensional space. If viewed in this way, this equality can be analyzed through a series of steps. The use of the orthogonal matrix V^T is based primarily on the rotation of the unit sphere of *n*-dimensional space by rotating the natural base into the base of the principal axes v_i. The diagonal matrix Σ stretches or contracts the sphere along the major axes into the ellipsoid. At the same time, a rectangular diagonal matrix Σ incorporates a sphere from n − dimensional space into an ellipsoid in m − dimensional vector space. The orthogonal matrix *U* rotates the ellipsoid to new major axes u_j in to m − dimensional space. The dimension of the space through which the ellipsoid extends is equal to the number of nonzero diagonal values of the matrix Σ [12].

There is a neat way to remove U and see V by itself. Multiply A^T times A. In this case practically SV decomposition matrix is closely related to the Gram matrix A^TA, and the equation was obtained as in

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T \Sigma V^T \tag{4}$$

U^TU disappears because it equals I. Multiplying those diagonal Σ^T and Σ gives σ_i^2 values. That leaves an ordinal diagonalization of the crucial symmetric matrix A^TA, whose eigenvalues are σ_i^2 values. The diagonal matrix $\Sigma^T\Sigma$ is orthogonally similar to the Gram matrix A^TA [12]. Therefore, the eigenvalues σ_i^2 of the matrix $\Sigma^T\Sigma$ are at the same time the eigenvalues of the matrix A^TA [13]. There are many properties and attributes of SVD. Some of the most important SVD properties and attributes in terms of application in the field of image processing are numbered bellow.

- singular values of $\sigma_1, \sigma_2, \ldots, \sigma_n$ are uniquely defined, but the matrices *U* and *V* are not,
- since $A^TA = V\Sigma^T\Sigma V^T$, so *V* diagonalizes A^TA, it follows that the v_j s are the eigenvector of A^TA,
- since $AA^T = U\Sigma\Sigma^TU^T$, so it follows that *U* diagonalizes AA^T and that the u_i 's are the eigenvectors of AA^T
- if *A* has rank of *r* then v_j, $v_j, \ldots,$ v_r form an orthonormal basis for range space of A^T, $R(A^T)$, and u_j, u_j, …, u_r form an orthonormal basis for range space *A*, *R(A)*.
- the rank of matrix *A* is equal to the number of its nonzero singular values,
- the eigenvalues of the A^TA matrix are also the squared singular values of the matrix A.

IV. Application Of SVD In Agricultural Image Compression

SVD can be applied to multiple problems in the domain of digital image processing. Observed from the angle of processing of images that represent agricultural areas, the challenge of image compression stands out. The need for image compression stems from the fact that most images that are obtained both from the air and from the ground have high-resolution, so they take up a large amount of memory. This is especially noticeable if a large number of images need to be stored for a long period in order to create a database of images. As part of the conducted research, the application of SVD practically represents the pre-processing of images obtained by recording agricultural plantations from the air. Initially, the images are collected for the development of a system for the recognition of the occurrence of diseases and pests and predicting the time of application of agrotechnical and chemical measures. As this task requires a significant number of high-resolution images, there is a need to compress them before storing them for further use. In addition to reducing memory usage, the use of compressed images further simplifies the process of their processing and use. The idea was that each of the selected images intended for use in the performed experiment should be represented in the form of a matrix, which can be done with a mathematical approach. Practically observed from the angle of mathematics and possible representation of the digital image, each digital image can be represented as matrix of pixel values.

If black and white images are observed, each little image element or “pixel” has a gray scale number between black and white. In the case of color pictures it has three numbers. Based on that, image compression deals with the problem of reducing the amount of data required to represent a digital image. When an image is SVD transformed, it is not compressed, but the data take a form in which the first singular value has a great amount of the image information. This property allows that only a few singular values need to be used in order to represent the image with little differences from the original.

Image compression using SVD was implemented within the MATLAB software package based on the following procedure. Decomposition of matrix A to the $U\Sigma V^T$ product represents an approximation of the matrix A using much smaller values compared to the original matrix.

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots + \sigma_r u_r v_r^T + 0 u_{r+1} v_{r+1}^T + \ldots \quad (5)$$

As singular values are always greater than zero, adding dependent members, where singular values are equal to zero, has no effect on the image. In the end, we get the equation with the members:

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots + \sigma_r u_r v_r^T \quad (6)$$

Furthermore, the obtained matrix can be further approximated by omitting the singular members of matrix A. As the singular values are sorted in descending order, the last members have the least influence on the final image. This is how the size of the memory space needed to store the newly created image is reduced. The closest matrix of rank k is obtained by truncating those sums after the first k terms:

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots + \sigma_k u_k v_k^T \quad (7)$$

If we observe the compression ratio, it can be calculated as in:

$$R = \frac{nk + k + mk}{nm} * 100 \quad (8)$$

Where R is the compression percentage, k is the chosen rank for truncation; m and n are the number of rows and columns in the image respectively. The calculation of the compression coefficient as well as the size of the compressed image is shown based on one selected representative image. This image was taken over the field with the use of a drone. Compression results obtained after the execution are displayed in the appendix on the Fig 3. The resolution of the original image is 3840 x 2160, the number of pixels of the uncompressed image is 8294400, where 3840 represents the vertical and number 2160 the horizontal number of pixels of the original image. If k = 20 which can be viewed as the number of iterations equation 7 will look like:

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots + \sigma_{20} u_{20} v_{20}^T \quad (9)$$

Based on the previous equation each of u_i contains 648 components, each of v_i contains 480 components, and how each σ_i is the scalar that represents 1 component. If we denote the compressed image with N_K, the number of pixels of the compressed image can be obtained as in:

$$N_k = 20 * (3840 + 2160 + 1) = 120020 \quad (10)$$

The compression ratio of image can be calculated as follows:

$$R = \frac{2160 * 20 + 20 + 3840 * 20}{2160 * 3840} * 100 = 1.45\% \quad (11)$$

Finally, after 20 iterations we got approximately the same image, which is 1.45% of the compressed image, while after 150 iterations we got a good compressed image, which is 10.85%. Compared to a compressed image with 20 iterations, an image with 150 iterations looks visually much better with noticeable sharpness and detail while still taking up little storage space. In the Fig. 2 error distribution between the compressed and the original image, which is taken as an example, is shown.

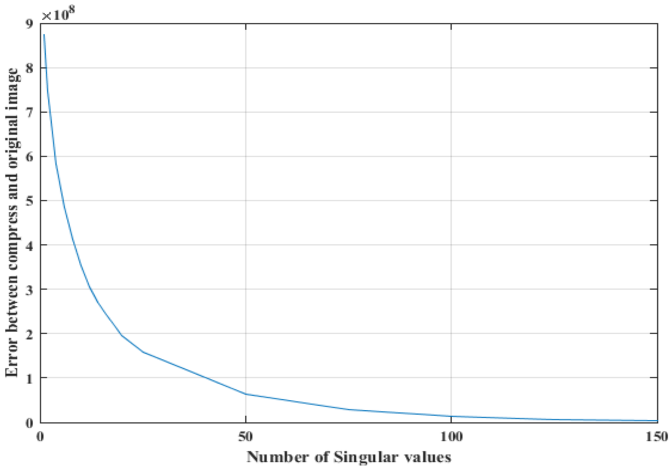

Fig. 2. Error distribution in compression process of selected image

As it can be seen from Fig. 2, as the number of singular values increases, the error decreases. When working with black and white images, experimental results have shown that less iteration is required in order to obtain adequate quality of compressed image.

If we look at the images given in Fig. 3, it is noticeable that with an initially small number of iterations, the quality of the compressed image decreases. When more than 75 singular values are used, the quality of the compressed image is almost identical to the original image, while the size of the compressed image is reduced by half compared to the original. As these are images whose purpose is to be used in the decision-making process, it is necessary for the image quality to be appropriate, which corresponds to a larger number of singular values.

The original image takes up 5.43MB of space, while the space required to store compressed images ranges from 888KB for an image obtained using 20 singular values to 1.35MB for an image obtained using 150 singular values.

V. Conclusion

Innovations in the field of digital image creation have created the possibility of creating high-resolution images. These images represent a significant source of information. However, high-resolution images take up significant memory space. If it is a large number of digital images, as is the case with images created for the needs of agriculture, where large areas are photographed by dividing the pre-defined quarters, it is necessary to process and compress the images obtained in this way. Due to the importance of information, compression must not reduce the quality of the image itself.

In this paper, we presented the fact that mathematical methods like SVD can be applied to image compression by looking at each of its pixels. With the use of this method we were able to compress both black and white and color images, obtained by aerial photography. In this way, the pre-processing of the images needed within the larger system for monitoring agricultural land was performed, which enabled easier storage and use of these images.

In addition to the use of both black and white, and color images in compression, the detection domain of the same or similar objects can also be used. This application can be important in the field of monitoring the situation on agricultural land. The use of SVD in object detection allows the comparison of the photographed object with pre-selected objects within the database. It is practically necessary to create a database of objects based on which the comparison will be made. Applied to the domain of agriculture, SVD can be used in estimating the number of weed communities on agricultural land, detection and marking of areas in fruit crops, detection and marking of uncultivated land, forests, underwater areas, etc. The starting point for detecting objects using SVD are certainly quality digital images. The implementation of the application of SVD in order to compare objects on compressed images in the database and new objects on newly obtained images is an idea for future research. This method can provide detection of pathogens and pests on sown crops.

Appendix

Fig. 3. Example of image compression with singular values used {(original image, 20), (25,50), (75,100), (125, 150)} respectively

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