Two-dimensional Green's function for a truncated wedge

Dragan Filipović, Tatjana Dlabač and Vladan Durković

***Abstract*—In this paper a closed form expression for the two-dimensional Green's function for a truncated wedge is derived. The method of separation of variables in Laplace's equation is used to get the function in the form of an infinite series of suitible harmonics and then the series is summed up in a closed form. Based on the obtained result the capacitance per unit length of a thin line conductor inside the wedge is evaluated.**

***Index Terms*—Green's function, truncated wedge, capacitance per unit length.**

I. Introduction

GREEN'S function is a powerful tool for solving boundary value problems for Poisson's equation (and other partial differential equations). Once this function is found for a given domain (which may be bounded or unbounded), the potential inside the domain is determined by integration, for an arbitrary charge distribution.

A widely used method for constructing Green's function is separation of variables [1]-[3] in the corresponding partial differential equation, which gives the function in the form of an infinite series or an improper integral with appropriate harmonics, depending on the sign of the separation constant.

In the two-dimensional case, it is possible to use the method of conformal mappings [4]. Also, in some particular cases, the method of images may be applied [1]-[3].

In this paper we use separation of variables in Laplace's equation in cylindrical coordinates to obtain the Green's function for a truncated wedge. The function is obtained in the form of an infinite series of properly chosen harmonics. It turns out that this series can be summed up in a closed form.

II. Derivation of Green's function for a truncated wedge

Fig. 1 shows the cross section of a bounded two-dimensional domain having the form of a wedge with angle α, truncated by a circular arc of radius R. By defintion, the Green's function is defined as the potential at the point (r,θ) inside the domain, created by a unit line charge at the point (r',θ'), provided the boundary of the domain is at zero potential.

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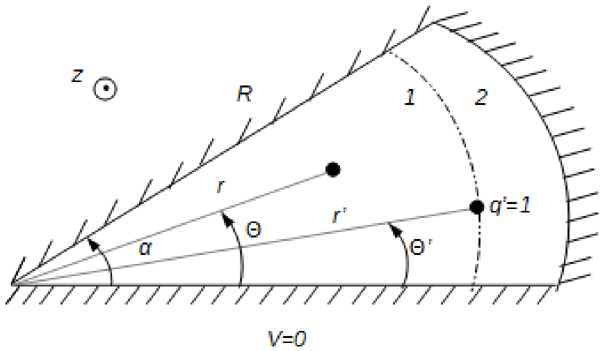

Fig. 1.

In other words, for fixed r' and θ' the Green's function

$$\boldsymbol{G(r,r')} \equiv \boldsymbol{V}(\boldsymbol{r}) \quad (\mathbf{r}=(r,\theta),\mathbf{r}'=(r',\theta'))$$

should satisfy Laplace's equation.

$$\Delta_r G(\boldsymbol{r},\boldsymbol{r}') = 0$$

everywhere inside the domain, except at the point $r=r'$, $\theta=\theta'$, where it must exhibit a singularity. Next, the Green's function has to meet the boundary condition

$$G\big|_S = 0$$

where S is the boundary of the considered domain, defined by $\theta=0$, $\theta=\alpha$ and $r=R$.

To determine the potential (i.e. the Green's function at an arbitrary point $r=(r,\theta)$), it is convenient to divide the domain into two subdomains 1 and 2, defined by $r<r'$ and $r'<r\leq R$, respectively, as shown in Fig.1.

Separation of variables in Lapalce's equation in cylindrical coordinates gives harmonics $r^{\pm k}$, $\cos k\theta$ and $\sin k\theta$. The boundary conditions for $\theta=0$ and $\theta=\alpha$ are satisfied if we choose the sine function with $k=n\pi/\alpha$. Therefore, we may write for the potential in subdomain 1.

$$V_1(r,\theta) = \sum_{n=1}^{\infty} A_n \cdot \left(\frac{r}{R}\right)^{\frac{n\pi}{\alpha}} \cdot \sin\frac{n\pi\theta}{\alpha},\ r<r' \tag{1}$$

where A_n (n=1,2,3...) are coefficients to be determined.

For subdomain 2 we must use a suitible combination of harmonics $r^{\pm n\pi/\theta}$ to satisfy the boundary condition $V_2=0$ for $r=R$.

This is achieved if we put

Proceedings of 4th International Conference on Electrical, Electronics and Computing Engineering, IcETRAN 2017, Kladovo, Serbia, June 05-08, ISBN 978-86-7466-692-0

pp. API1.2.1-3

$$V_2(r,\theta)=\sum_{n=1}^{\infty}B_n\left[\left(\frac{r}{R}\right)^{\frac{n\pi}{\alpha}}-\left(\frac{r}{R}\right)^{-\frac{n\pi}{\alpha}}\right]\sin\frac{n\pi\theta}{\alpha},\ r'<r\leq R \quad (2)$$

with some unknown coefficinets B_n.

Continuity of the potential at $r=r'$ requires:

$$A_n\left(\frac{r'}{R}\right)^{\frac{n\pi}{\alpha}}=B_n\left[\left(\frac{r'}{R}\right)^{\frac{n\pi}{\alpha}}-\left(\frac{r'}{R}\right)^{-\frac{n\pi}{\alpha}}\right]$$

whence

$$B_n=\frac{\left(\frac{r'}{R}\right)^{\frac{n\pi}{\alpha}}}{\left(\frac{r'}{R}\right)^{\frac{n\pi}{\alpha}}-\left(\frac{r'}{R}\right)^{-\frac{n\pi}{\alpha}}}A_n. \quad (3)$$

So far, we have not used the unit line charge which determines the potential singularity; it will be convenient to use the δ-function formalism. To this end, we “spread” the unit charge $q'=1$ over the surface $r=r'$ with surface density

$$\sigma=\frac{1}{r'}\cdot\delta(\theta\text{-}\theta').$$

This surface density, when integrating over the surface $r=r'$ gives the total unit charge

$$\int_s\sigma ds=\int_0^{\alpha}\frac{1}{r'}\cdot\delta(\theta\text{-}\theta')\cdot r'd\theta=\int_0^{\alpha}\delta(\theta\text{-}\theta')d\theta'=1=q'.$$

Now, it remains to make use of the boundary condition for the normal components of the electric field on the surface $r=r'$.

$$\left.\left(\frac{\partial V_1}{\partial r}-\frac{\partial V_2}{\partial r}\right)\right|_{r=r'}=\frac{\sigma}{\varepsilon_0}=\frac{1}{\varepsilon_0\cdot r'}\cdot\delta(\theta\text{-}\theta'). \quad (4)$$

Using (1)-(3), from (4) we obtain

$$2\cdot\frac{\pi}{\alpha}\sum_{n=1}^{\infty}\frac{n\cdot A_n\cdot\sin\frac{n\pi\theta}{\alpha}}{\left(\frac{r'}{R}\right)^{-\frac{n\pi}{\alpha}}-\left(\frac{r'}{R}\right)^{\frac{n\pi}{\alpha}}}=\frac{1}{\varepsilon_0}\cdot\delta(\theta\text{-}\theta'). \quad (5)$$

To obtain the coefficients A_n, we multiply both sides of (5) by $\sin(m\pi\theta/\alpha)$ and integrate with respect to θ from $\theta=0$ to $\theta=\alpha$. Since the functions $\sin(k\pi\theta/\alpha)$ form an orthogonal set over the segment [0, α], all the terms in the summation in (5) vanish, except the one for $n=m$. Therefore:

$$\frac{2\frac{m\pi}{\alpha}\cdot\frac{\alpha}{2}A_m}{\left(\frac{r'}{R}\right)^{-\frac{m\pi}{\alpha}}-\left(\frac{r'}{R}\right)^{\frac{m\pi}{\alpha}}}=\frac{1}{\varepsilon_0}\cdot\sin\frac{m\pi\theta'}{\alpha} \quad (6)$$

where we also used the fact that

$$\int_0^{\alpha}\delta(\theta\text{-}\theta')\cdot\sin\frac{m\pi\theta}{\alpha}d\theta=\sin\frac{m\pi\theta'}{\alpha}.$$

From (6), the coefficients A_n are

$$A_n=\frac{1}{\varepsilon_0\pi n}\cdot\sin\frac{n\pi\theta'}{\alpha}\left[\left(\frac{r'}{R}\right)^{-\frac{n\pi}{\alpha}}-\left(\frac{r'}{R}\right)^{\frac{n\pi}{\alpha}}\right]. \quad (7)$$

Finally, by using (7) and (3), the potentials (1)-(2) in the two subdomains become

$$V_1(r,\theta)=\frac{1}{\varepsilon_0\pi}\times$$
$$\times\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{r'}\right)^{\frac{n\pi}{\alpha}}-\left(\frac{r\cdot r'}{R^2}\right)^{\frac{n\pi}{\alpha}}\right]\sin\frac{n\pi\theta'}{\alpha}\sin\frac{n\pi\theta}{\alpha},\ r<r' \quad (8)$$

$$V_2(r,\theta)=\frac{1}{\varepsilon_0\pi}\times$$
$$\times\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r'}{r}\right)^{\frac{n\pi}{\alpha}}-\left(\frac{r\cdot r'}{R^2}\right)^{\frac{n\pi}{\alpha}}\right]\sin\frac{n\pi\theta'}{\alpha}\sin\frac{n\pi\theta}{\alpha},r'<r\leq R \quad (9)$$

Expressions (9)-(10) constitute the Green’s function $G(r,r')$ for the considered domain.

The summations in (8)-(9), although seemingly rather complicated, can be summed up in a closed form by using [5]

$$\sum_{n=1}^{\infty}\frac{A^n}{n}\sin nx\cdot\sin ny=\frac{1}{4}\cdot\ln\frac{A^2-2A\cos(x+y)+1}{A^2-2A\cos(x-y)+1},\ |A|<1$$

Hence,

$$V_1(r,\theta)=\frac{1}{4\pi\varepsilon_0}\cdot\ln\left[\frac{\left(\frac{r}{r'}\right)^{\frac{2\pi}{\alpha}}-2\left(\frac{r}{r'}\right)^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}(\theta+\theta')+1}{\left(\frac{r}{r'}\right)^{\frac{2\pi}{\alpha}}-2\left(\frac{r}{r'}\right)^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}(\theta\text{-}\theta')+1}\times\right.$$
$$\left.\times\frac{\left(\frac{r\cdot r'}{R^2}\right)^{\frac{2\pi}{\alpha}}-2\left(\frac{r\cdot r'}{R^2}\right)^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}(\theta\text{-}\theta')+1}{\left(\frac{r\cdot r'}{R^2}\right)^{\frac{2\pi}{\alpha}}-2\left(\frac{r\cdot r'}{R^2}\right)^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}(\theta+\theta')+1}\right],\ r<r' \quad (10)$$

The potential V_2, given by (9), can be obtained from (10) by interchanging r and r'. But, as it can be readily verified, this interchange leaves (10) invariant, so formula (10) is also valid for V_2 and consequently for the Green's function.

The Green's function for an infinite wedge can be obtained from (10), if we let $R\to\infty$. It is

$$V(r,\theta)=\frac{1}{4\pi\varepsilon_0}\cdot\ln\left[\frac{\left(\frac{r}{r'}\right)^{\frac{2\pi}{\alpha}}-2\left(\frac{r}{r'}\right)^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}(\theta+\theta')+1}{\left(\frac{r}{r'}\right)^{\frac{2\pi}{\alpha}}-2\left(\frac{r}{r'}\right)^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}(\theta-\theta')+1}\right].$$

III. Capacitance of a thin line conductor inside the wedge

Let a thin line conductor of radius r_o, carrying charge q' per unit length be positioned at $r'=a$, $\theta'=\beta$. We assume that the conductor radius is very small compared to the distances from the conductor to the wedge walls. The conductor potential can be found from (10), if we put $r'=a$, $\theta'=\beta$, $r=a-r_o$, $\theta=\beta$. Actually, we put $r=a-r_o$ only in denominator of the first fraction under the logarithm, elsewhere we may take $r=a$. Using the approximation

$$\left(\frac{a-r_o}{a}\right)^{\frac{2\pi}{\alpha}}-2\left(\frac{a-r_o}{a}\right)^{\frac{\pi}{\alpha}}+1=\left[\left(\frac{a-r_o}{r_o}\right)^{\frac{\pi}{\alpha}}-1\right]^2\approx\left(\frac{\pi r_o}{\alpha a}\right)^2.$$

we find from (10)

$$V_{cond}=\frac{q'}{4\pi\varepsilon_0}\cdot\ln\left[\frac{4(a\alpha)^2\sin^2\frac{\pi\beta}{\alpha}\left[\left(\frac{a^2}{R^2}\right)^{\frac{\pi}{\alpha}}-1\right]^2}{(\pi r_0)^2\left[\left(\frac{a^2}{R^2}\right)^{\frac{2\pi}{\alpha}}-2\left(\frac{a^2}{R^2}\right)^{\frac{\pi}{\alpha}}\cos\frac{2\pi\beta}{\alpha}+1\right]}\right]$$

and the conductor capacitance per unit length is

$$C=\frac{q'}{V_{cond}}=\frac{2\pi\varepsilon_0}{\ln\dfrac{2a\alpha\sin\frac{\pi\beta}{\alpha}\left[1-\left(\frac{a^2}{R^2}\right)^{\frac{\pi}{\alpha}}\right]}{\pi r_0\sqrt{\left(\frac{a^2}{R^2}\right)^{\frac{2\pi}{\alpha}}-2\left(\frac{a^2}{R^2}\right)^{\frac{\pi}{\alpha}}\cos\frac{2\pi\beta}{\alpha}+1}}}.$$

IV. Conclusion

In this paper we derived Green's function for a truncated wedge, by using separation of variables in Laplace's equation in cylindrical coordinates. The obtained infinite series for the function is summed up in a closed form. Also, the capacitance per unit length of a thin line conductor inside the wedge is obtained. The applied method can be used for other shapes of truncated wedges.

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